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A Note about Torsional Rigidity and Euclidean Moment of Inertia of Plane Domains

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Abstract—Denote by $\mathbf{P}(G)$ the torsional rigidity of a simply connected plane domain G, and by $\mathbf{I}_2(G)$ the Euclidean moment of inertia of G. In 1995 F.G. Avkhadiev proved that $\mathbf{P}(G)$ and $\mathbf{I}_2(G)$ are comparable quantities in sense of Pólya and Szegö. Moreover, it was shown that the ratio $\mathbf{P}(G)/\mathbf{I}_2(G)$ belongs to the segment $[1, 64]$. We investigate the following conjecture $\mathbf{P}(G) \geq 3\mathbf{I}_2(G)$, where G is a simply connected domain. We prove that the conjecture is true for polygonal domains circumscribed about a circle. For convex domains we show sharp isoperimetric inequalities, which justify the conjecture, in particular, we prove that $\mathbf{P}(G) > 2\mathbf{I}_2(G)$. Some aspects of approximate formulas for $\mathbf{P}(G)$ are also discussed.

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1. INTRODUCTION AND SOME CONJECTURES

In analytical studies of steady field problems of the theory of elasticity [1] and microfluid mechanics [2] a boundary value problem

$$\triangle \mathrm{u} = -2 \ \text{ in } G, \quad \mathrm{u} = 0 \ \text{ on } \partial G \tag{1}$$

is solved for complex-shaped domains G, here ∂G is the boundary curve of G. A classical result of the theory in partial differential equations is that there is a unique solution of the boundary value problem (1) for a simply connected domain G. Isoperimetric estimates in these solutions (see, for instance, [3, 4]) require evaluation of the functional

$$\mathbf{P}(G)^{1)} := 2\int_G \mathrm{u}(x, G)\mathrm{dA}. \tag{2}$$

This functional is called the *torsional rigidity* of G in the theory of elasticity, and the *flow rate* in hydrodynamics of pipe flows.

Investigations of the physically important boundary value problem (1), and the physical functional (2) go back to works of B. de Saint-Venant [5], and Lord Rayleigh (see [3, 6]). Estimates of $\mathbf{P}(G)$, using different geometrical or (and) physical characteristics of G, intensively studied by mathematicians since the middle of XX century, when their interest was stimulated by the famous monograph of G. Pólya and G. Szegö [3]. Then a number of works [6] is growing rapidly in this area. There are a number of isoperimetric inequalities [6] for $\mathbf{P}(G)$ in this branch of mathematical physics.

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1)Throughout the paper we will use the bold face for notations of functionals.